# Lecture 13

$G$ connected cplx semisimple.
$P \subset G$ parabolic $\quad K \subset G$ compact real form $\Leftrightarrow$ max cpt subgrp

$G/P \cong K/_{K\cap P}$ compact complex manifold w/
Transitive holomorphic action of $G$
Transitive smooth action of $K$ $\rightsquigarrow$ $K$-invariant Riem metric.

There are many choices for $K$, all conj. Param by $G/K$, the symmetric space of $G$, which is another important player.

$G = SL_2\mathbb{C}$ $\quad K = SU(2)$ $\quad P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $\quad G/P = \mathbb{CP}^1 \cong S^2$

$G/K \cong \mathbb{H}^3 \cong$ open 3-ball, $G$ acts by $\mathrm{Isom}(\mathbb{H}^3)$ on interior.
$G$ acts by Möbius trans on $\mathbb{CP}^1$.
Viewing $G/P$ as $K/T$ gives $\mathbb{CP}^1$ a spherical metric.
An element of $G$ preserves this metric iff it fixes the center of the Ball model of $\mathbb{H}^3$.

## Topology of a flag variety

**Thm.** $G/P$ has a CW complex structure where the cells have even dimensions ($\cong \mathbb{C}^k$) and the set of cells is in bijection with $W(G)$. There is one 0-cell ($e \in W$) and $\mathrm{rk}(G)$ 2-cells ($s_\alpha \in W$, $\alpha \in \Delta$).

This is a consequence of the Bruhat decomposition. $G/B$ first!

Before proving it, note it follows that all generalized $\mathbb{C}$ flag varieties are simply connected.

**Thm.** (H.C. Wang 1954) If $M$ is a compact, simply connected complex manifold that admits a transitive holomorphic group action, then $\mathrm{Aut}(M) = \{f: M \to M \text{ bihol}\}$ is a semisimple Lie group and $\mathrm{Stab}_{\mathrm{Aut}(M)}(p)$ is a parabolic subgroup. In particular $M$ is a complex flag variety.

**Warning**: $G$ not unique... $\mathbb{CP}^3 = SL_4\mathbb{C}/P = Sp(4,\mathbb{C})/Q$
$\mathrm{Aut}(Sp(4,\mathbb{C})/Q) = SL_4\mathbb{C}$.

For $g \in G$ let $BgB = \{b_1 g b_2 \mid b_i \in B\}$, a double coset.

$\boxed{\text{Bruhat Decomp}}$

**Theorem**. $G = \bigcup_{w \in W} BwB$ where $W = N(H)/H$ $\quad$ $\mathrm{Lie}(H) = \mathfrak{h} =$ Cartan

well-def as $H \subset B$.

$H \cong (\mathbb{C}^*)^{\mathrm{rk}(G)}$ maximal "torus". (Any two max tori are conj in $G$ or in $B$

AND: $N_B(H) = H$.

We'll depend on one tricky Lie group fact:

**Thm**: Let $B, B' \subset G$ be Borel subgps, $G$ complex semisimple. Then $B \cap B'$ contains a maximal alg torus.
(Further, if $B \supset H$ then $\exists!$ Borel $B'$ s.t. $B \cap B' = H$.

Write $\mathfrak{g} = \underbrace{\mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+}_{\text{root spaces}}$ with $H = N_G(\mathfrak{h})^\circ$, $\mathrm{Lie}(B) = \mathfrak{h} \oplus \mathfrak{g}^+$

Then $B^- = N_G(\mathfrak{g}^- \oplus \mathfrak{h})$.) $\quad$ see Borel, Linear Alg Groups Cor 14.13 on p196. $\square$

**Pf. of Bruhat**. Fixed: $H \subset B \subset G$.
Let $g \in G$. Want to find $w \in W$ s.t. $g \in BwB$.
Let $H' \in B \cap gBg^{-1}$.
Since $H, H' \subset B$ $\exists$ $b \in B$ s.t. $H' = bHb^{-1}$. $\rightsquigarrow$ $H = b^{-1}H'b$
Since $gHg^{-1}, H' \subset gBg^{-1}$, $\exists$ $c \in gBg^{-1}$ s.t. $H' = c(gHg^{-1})c^{-1}$
$c = gb'g^{-1}$ so $H' = gb'H(gb')^{-1}$. $\quad$ Here $b' \in B$.
Thus $H \xrightarrow{\text{conj } gb'} H' \xrightarrow{\text{conj } b^{-1}} H$, i.e. $\overbrace{b^{-1}gb'}^{\tilde{f}(g)} = \nu \in N(H)$.

Hence $g \in B v B$. Now $v$ not well def but $b, b'$ both well def up to $h \in N_B(H) = H$ so any $v'$ has form

$$h v h' = v \underbrace{h'' h'}_{} \in vH.$$

$v$ normalizes $H$

so the coset $v \in N(H)/H$ is well defined.

$\Rightarrow G = \bigcup_{w \in W} BwB$ set-theoretically, disjoint. $\square$

**Cor.** Let $x_0 \in G/B$ denote $eB$. Then $G/B = \bigcup_{w \in W} Bwx_0$.

That is, $B \curvearrowright G/B$ has finitely many orbits.

Denote $C_w = Bwx_0 \subset G/B$

**Thm.** For $w \in W$ let $\ell(w) = \#\{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\} = \#$ inversions.
Then $C_w$ is an embedded submfld of $G/B$ iso to $\mathbb{C}^{\ell(w)}$.

Pf sketch. Let $\Phi_w^- = \Phi^- \cap w(\Phi^+)$. So $\ell(w) = \#(\Phi_w^-)$

Then $n_w$ cog def by $\bigoplus_{\alpha \in \Phi_w^-} \mathfrak{g}_\alpha$ has associated Lie group $N_w$
$\cong \mathbb{C}^{\ell(w)}$. The map $N_w \longrightarrow G/B$ is an embedding.

$$a \longmapsto \underset{\substack{\cap \\ B}}{waw^{-1}} w x_0$$

$\square$

$C_w$ is called the Schubert cell associated to $w$.

e.g. $C_e$ is just the point $x_0$.

**Thm.** $\overline{C_w} = \bigcup_{x \in \underline{w}} C_x$ where $\underline{w} = \{x \in W \mid \Phi_x^- \subset \Phi_w^-\}$

Say $w' \leq w$ if $\Phi_{w'}^{-} \subset \Phi_{w}^{-}$. Called Bruhat order on $W$.

<u>Example</u>. $S_3 = W(SL_3\mathbb{C})$.